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SOME PROPERTIES OF THE SOLUTIONS OF PLANE THERMOELASTIC PROBLEMS OF THE RATIONAL REINFORCEMENT OF COMPOSITE STRUCTURES†

Yu. V. NEMIROVSKII and A. P. YANKOVSKII

Novosibirsk

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The problem of the reinforcement of plane structures with uniformly stressed fibres of constant cross-section taking the thermal effects into account is formulated. The properties of the solution of this problem, which are determined by the heat conduction equation and the condition of constancy of the cross-sections of the fibres, are investigated. The effect of the temperature field on the structure of the reinforcement is analysed.

A formulation of the problem of the rational reinforcement (RR) of plane composite structures with two families of high-modulus uniformly stressed fibres of constant cross-section taking thermal effects into account has been given previously in [1]. The corresponding system of equations possesses a number of singularities which make the properties of its solutions difficult to investigate in the general case. Some properties of the solutions of RR problems which have been formulated with less-restrictive constraints than in [1] and take account of the temperature field are investigated below.

1. INITIAL EQUATIONS

The complete system of equations which describes the behaviour of plane structures, which are loaded and reinforced by N families of fibres in the plane of the structure, has the form (the materials of the matrix and the fibres are assumed to be isotropic and their behaviour is assumed to be linearly elastic) the equilibrium equations and the differential Cauchy relations

$$\sigma_{1i,1}+\sigma_{i2,2}=-b_i,\quad i=1,2 \tag{1.1}$$

$$\varepsilon_{ij}=\tfrac{1}{2}(u_{i,j}+u_{j,i})/2,\quad i,j=1,2 \tag{1.2}$$

the Duhamel–Neumann relations [2]

$$\sigma_{ij}=a\sigma_{ij}^c+\sum_k \sigma_k\omega_k l_{ki}l_{kj},\quad i,j=1,2 \tag{1.3}$$

$$\sigma_{ii}^c=Ea_1(\varepsilon_{ii}+\nu\varepsilon_{jj}-\alpha_c(1+\nu)\theta),\quad \sigma_{ij}^c=Ea_2\varepsilon_{ij}$$

$$j=3-i,\ i=1,2;\ a_1=1/(1-\nu^2),\ a_2=1/(1+\nu)$$

$$l_{k1}=\cos\alpha_k,\quad l_{k2}=\sin\alpha_k,\quad a=1-\Omega,\quad \Omega=\sum_k\omega_k$$

the condition for uniform tensioning of the fibres

$$\sigma_k=E_k\varepsilon_k=\text{const},\quad k=1,2,\ldots,N \tag{1.4}$$

$$\varepsilon_k=\varepsilon_{11}l_{k1}^2+\varepsilon_{22}l_{k2}^2+2\varepsilon_{12}l_{k2}-\alpha_{ak}\theta$$

and the condition for the cross-sections of the fibres to be constant [1]

$$(\omega_k l_{k1})_{,1}+(\omega_k l_{k2})_{,2}=0,\quad k=1,2,\ldots,N \tag{1.5}$$

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The existence of thermal action not only has an effect on the structure of the rational reinforcement but the temperature field itself in the structure depends very much on the RR parameters, which leads to the related problems of determining the structure of the reinforcement and the temperature field. In solving such a complex problem as the rational reinforcement of structures with uniformly tensioned fibres, it is natural to consider those models of heat conduction which describe the basic thermal properties of a composition and, at the same time, have the simplest form for the subsequent analysis.

In the case of a unidirectionally reinforced material, the integral longitudinal thermal conductivity is well described to a first approximation by the law for a simple mixture and the determination of the integral transverse thermal conductivities requires the inclusion of a special mathematical apparatus [3]. On the basis of hypotheses, similar to those used in [4] when deriving the components of the compliance tensor of a unidirectionally reinforced elastic material, we obtain that a quantity, which is the inverse of the integral transverse thermal conductivity, is determined using the rule for mixtures of quantities which are the inverses of the thermal conductivities of the fibre and the binder. (The averaged thermal conductivities formed in this way are coefficients of positive definite quadratic form.) In the case of a composite material reinforced with N families of fibres in parallel planes, we determine the integral thermal conductivities by averaging (proportional to the functions ω_k) the thermal conductivities of the unidirectionally reinforced layers. The thermal conductivities of the whole packet (Λ_{ij}) averaged in this way will be coefficients of positive definite quadratic form if the physical constraints

$$0<\omega_k \quad (k=1,2,\ldots,N), \quad \Omega<1 \tag{1.6}$$

are satisfied and the plane steady-state heat conduction equation takes the form

$$(\Lambda_{11}\theta_{,1}+\Lambda_{12}\theta_{,2})_{,1}+(\Lambda_{21}\theta_{,1}+\Lambda_{22}\theta_{,2})_{,2}=-w \tag{1.7}$$

$$\Lambda_{ij}=\Omega^{-1}\sum_k \omega_k\{[\Omega(\lambda_k-\lambda_c)+\lambda_c]l_{ki}l_{kj}+(-1)^{i+j}l_{kl}l_{km}\lambda_k\lambda_c[\Omega(\lambda_c-\lambda_k)+\lambda_k]^{-1}\},$$

$$l=3-i, \quad m=3-j, \quad i,j=1,2$$

The following notation is used in Eqs (1.1)–(1.7): σ_{ij} and ε_{ij} are the components of the averaged stresses and strains, σ_{ij}^c are the components of the stresses in the binder, σ_k and ε_k are the stress and the mechanical deformation in the reinforcing of the kth family, ω_k is the intensity of the reinforcement by a fibre of the kth family, α_k is the angle between the tangent to the trajectory of a fibre of the kth family and the x_1 axis, a is the intensity of the reinforcement by layers of the binder (with certain additional assumptions, it can be taken that a = const, as was done in [1], for example); E and E_k are Young's moduli for the materials of the matrix and the reinforcement of the kth family, ν is Poisson's ratio of the binder, u_i and b_i are the components of the displacement and the mass distribution of the load along the directions x_i of a rectangular Cartesian system of coordinates, where $b_i = (a\rho_c + \sum_k \omega_k\rho_k)\, F_i$ ($i = 1, 2$), ρ_c and ρ_k are the mass densities of the materials of the binder and the fibres of the kth family, F_i are the components of the specific load distribution which acts per unit mass, α_c and α_{ak} are the coefficients of linear thermal expansion of the binder and the reinforcement of the kth family, λ_c and λ_k are the thermal conductivities of the matrix and the reinforcement of the kth family, θ is the deviation of the actual temperature of the structure from the temperature in its natural state, and w is the intensity of the internal heat sources; here and henceforth summation is carried out with respect to an indicated index from 1 to N if no limits are specified and a subscript after a comma denotes differentiation with respect to the corresponding coordinate x_i.

The static boundary conditions

$$\begin{aligned}&\sigma_{11}n_1^2+\sigma_{22}n_2^2+2\sigma_{12}n_1n_2=p_n(s)\\&(\sigma_{22}-\sigma_{11})n_1n_2+\sigma_{12}(n_1^2-n_2^2)=p_\tau(S)\end{aligned} \tag{1.8}$$

can be specified on one part of the contour of the structure (Γ_p) and the kinematic conditions

$$u_i(\Gamma_u)=u_{i0}(s), \quad i=1,2 \tag{1.9}$$

can be specified on another part (Γ_u). The thermal boundary conditions

$$\chi_0[(\Lambda_{11}\theta_{,1}+\Lambda_{12}\theta_{,2})n_1+(\Lambda_{21}\theta_{,1}+\Lambda_{22}\theta_{,2})n_2+q_0]+\chi_1(\theta-\theta_0)=0 \tag{1.10}$$

can be specified over the whole of the contour Γ. (Each of the conditions (1.8) and (1.9) can also be specified over the whole of the contour Γ which bounds the domain G which is occupied by the structure in a plan view.) Here, $n_1 = \cos\beta$, $n_2 = \sin\beta$ where β is the angle specifying the direction of the outward normal to Γ, p_n and p_τ are the normal and shear stress in the contour, respectively, u_{i0} is the displacement in the contour, $q_0(s)$ is the heat flux across the surface which bounds the structure, $\theta_0(s)$ is the difference between the temperature of the surroundings on the contour and the temperature of the structure in a natural state, χ_0 and χ_1 are switching functions which enable one to select any type of thermal boundary conditions and s is a parameter which defines the contour.

Since Eqs (1.5) are consequences of the integral laws of conservation of the cross-sections of the fibres [1], the functions ω_k can only be specified on that part of the contour (Γ_ω) in which fibres occur in the structure. It is therefore necessary to choose the boundary conditions

$$\omega_k(\Gamma_\omega)=\omega_{nk}(s), \quad k=1,2,\ldots,N \tag{1.11}$$

in Γ_ω (not to be confused with the initial conditions in unsteady-state problems in mechanics).

2. THE SYSTEM OF RESOLVENTS

In order to reduce system (1.1)–(1.5), (1.7) and boundary conditions (1.8) to a resolvent form, it is necessary to substitute relation (1.2) into (1.3) and (1.4) and to substitute the result of this into (1.1) and (1.8), after which a closed system of equations and boundary conditions is obtained which contains the displacements u_i, the reinforcement parameters ω_k, α_k and the temperature θ as the unknowns

$$\begin{gathered}(-1)^i\sum_k \sigma_k\omega_k l_{kj}\partial_{sk}(\alpha_k)+E\Big\{aa_1[u_{i,ii}+\nu u_{j,ij}+\tfrac{1}{2}(1-\nu)(u_{j,ij}+u_{i,jj})]- \\ -a\alpha_c\theta_{,i}(1-\nu)^{-1}-\sum_k\{a_1[u_{i,i}+\nu u_{j,j}-\alpha_c(1+\nu)\theta]\omega_{k,i}+\tfrac{1}{2}a_2(u_{i,j}+u_{j,i})\omega_{k,j}\}\Big\}=-b_i \\ j=3-i, \quad i=1,\ 2\end{gathered} \tag{2.1}$$

$$\partial_{sk}(\alpha_k)+\omega_k\partial_{nk}(\alpha_k)=0, \quad k=1,2,\ldots,N \tag{2.2}$$

$$\partial_{sk}(u_1)\cos\alpha_k+\partial_{sk}(u_2)\sin\alpha_2-\alpha_{ak}\theta=\varepsilon_k=\text{const} \tag{2.3}$$

$$(\Lambda_{11}\theta_{,1}+\Lambda_{12}\theta_{,2})_{,1}+(\Lambda_{21}\theta_{,1}+\Lambda_{22}\theta_{,2})_{,2}=-w \tag{2.4}$$

where ∂_{sk} and ∂_{nk} are operators of differentiation along directions which are tangential and normal to the trajectory of the reinforcement of the kth family

$$\partial_{sk}(f)=f_{,1}l_{k1}+f_{,2}l_{k2}, \quad \partial_{nk}(f)=-f_{,1}l_{k2}+f_{,2}l_{k1} \tag{2.5}$$

and f is an arbitrary differentiable function.

Boundary conditions (1.8) take the form ($(x_1, x_2)\in\Gamma_p$)

$$\begin{gathered}\sum_k \sigma_k\omega_k\cos^2(\alpha_k-\beta)+Eaa_1[n_1^2(u_{1,1}+\nu u_{2,2})+n_2^2(\nu u_{1,1}+u_{2,2})+ \\ +(1-\nu)(u_{1,2}+u_{2,1})n_1n_2-\alpha_c\theta a_2^{-1}]=p_n(s) \\ \sum_k \sigma_k\omega_k\sin 2(\alpha_k-\beta)+Eaa_2[2n_1n_2(u_{2,2}-u_{1,1})+(n_1^2-n_2^2)(u_{1,2}+u_{2,1})]=2p_\tau(s)\end{gathered} \tag{2.6}$$

The remaining conditions (1.9)–(1.11) remain unchanged.

The system of resolvents (2.1)–(2.4) and the boundary conditions (1.10), (2.6) show that problems of determining the stress–strain state of a structure, the temperature field and the RR parameters are connected and they have to be solved together.

We will use the determinant method in [5] to determine the type of system (2.1)–(2.4). Since Eqs (2.3) do not contain higher derivatives of the unknown functions, the characteristic determinant of this system is identically zero. Hence, in order to clarify its type, we need to differentiate each equation of (2.3) by applying the operators ∂_{sk}, for example. The characteristic equation of the transformed system has the form

$$(\Lambda_{11}x_2'^2-2\Lambda_{12}x_2'+\Lambda_{22})\Bigg\{\tfrac{1}{2}E^2a^2a_1a_2(1+x_2'^2)^2+\tfrac{1}{2}Eaa_1a_2^{-1}\sum_k s_k\xi_k^3\eta_k-$$

$$-\sum_i\sum_{j>i}s_is_j\xi_i^2\xi_j^2\sin^2(\alpha_i-\alpha_j)+\sum_{n=1,2}(\sigma_{n2}^c-\sigma_{1n}^c x_2')\Bigg[\sum_j\omega_j\eta_j\Big(l_{jm}\Big[-\tfrac{1}{2}Eaa_1a_2^{-1}x_2'+$$

$$+(-1)^n\sum_k l_{kn}^2\xi_k^2 s_k\Big]-\Big[Eaa_1\varkappa_n+(-1)^n\sum_k l_{k1}l_{k2}\xi_k^2 s_k\Big]l_{jn}\Big)\Delta_j^{-1}\Bigg]\Bigg\}\xi_1\xi_2\ldots\xi_N=0,\quad m=3-n \qquad (2.7)$$

Here

$$\xi_k=l_{k2}-l_{k1}x_2',\quad \eta_k=l_{k1}+l_{k2}x_2',\quad \varkappa_1=1+\tfrac{1}{2}(1-\nu)x_2'^2$$

$$\varkappa_2=\tfrac{1}{2}(1-\nu)+x_2'^2,\quad s_k=\sigma_k\omega_k\Delta_k^{-1},\quad \Delta_k=l_{k2}\partial_{sk}(u_1)-l_{k1}\partial_{sk}(u_2)\ (k=1,2,\ldots,N)$$

and the derivative $x_2'(x_1)=dx_2/dx_1$ specifies the direction of the characteristic. The factors occurring in (2.7) after the braces indicate that the transformed system of resolvents has N real characteristics which coincide with the trajectories of the reinforcement. The trinomial in expression (2.7) in front of the braces only has complex roots x_2'.

Actually, it has been pointed out in Section 1 that, when the constraints (1.6) are satisfied, the coefficients Λ_{ij} accompanying the arbitrary reinforcement parameters are coefficients of a positive definite quadratic form and this means that the inequality

$$D=(-2\Lambda_{12})^2-4\Lambda_{11}\Lambda_{22}<0 \qquad (2.8)$$

is satisfied by them.

By virtue of this, the trinomial in (2.7) only has complex roots.

Consequently, the system of resolvents, when account is taken of thermal effects, has two complex characteristic directions. The expression enclosed in the braces in (2.7) is a fourth-order algebraic polynomial in x_2' which, depending on the values of the unknown functions and their derivatives, can have a different number of real roots and the system of resolvents (2.1)–(2.4) is therefore a quasilinear system of mixed-composite type [6].

3. SOME PROPERTIES OF THE SOLUTION OF SYSTEM (2.1)–(2.4)

At the present time, the theory of mixed-composite equations is still very incompletely developed [6] as regards the possibility of investigating the properties of the solution of system (2.1)–(2.4) analytically in general form. However, certain properties of the solution of this problem can be noted.

A negative value of the discriminant D means that Eq. (2.4) for θ, written in a divergent form for arbitrary reinforcement parameters α_k, ω_k ($1\le k\le N$), taking account of the constraints (1.6), satisfies the conditions of ellipticity [5]. This means that, when there are no internal heat sources ($w=0$), a function of the temperature θ can only attain its maximum and minimum values on the contour Γ. In particular, it follows from this that, when $w=0$ and when a constant value $\theta_0(\chi=0,\chi_1=1)$ or a zero heat flux $q_0=(\chi_0=1,\chi_1=0)$ is specified in boundary conditions (1.10) over the whole of the contour Γ, the function θ will also be constant over the whole of the structure and equal to θ_0. Consequently, the heat conduction problem may be considered to be solved, regardless of the solution of the RR problem although, as before, the structure of the rational reinforcement will depend very much on the level of heating or cooling of the structure.

We will now investigate some properties of functions of the intensity of the reinforcement ω_k, which satisfy the conditions for the cross-sections of the fibres to be constant (2.2). We assume that the fibres are stacked in the structure in some manner, that is, the trajectories of the reinforcement are known. Then, integrating (2.2) along the trajectories of the reinforcement, starting from the contour Γ_ω, on which initial conditions (1.11) are specified, we obtain

$$\omega_k = \omega_{ik} \exp\left(-\int_0^{l_k} \partial_{nk}(\alpha_k) dl_k\right), \quad k = 1, 2, \ldots, N \tag{3.1}$$

where l_k is the variable length of an arc along a chosen reinforcement trajectory of the kth family. It follows from equality (3.1) that the sign of the function ω_k is completely determined by the sign of the initial value of ω_k. Hence, if the functions ω_k are specified such that they satisfy the first N in equalities (1.6), these inequalities will hold for ω_k at all points of the structure. Moreover, if $\omega_k = 0$, the function ω_k will be equal to zero along the whole of the formal reinforcement trajectory. Consequently, when solving the rational reinforcement problem, it is only necessary to check that the last inequality of (1.6) is satisfied (violation of this inequality physically denotes "buckling" of the fibres from the reinforcement plane).

It turns out that the family of trajectories of fibres of constant cross-section cannot have an envelope.

To prove this, we use the following form of the condition for the cross-sections of the fibres to be constant [1]

$$\oint_L \boldsymbol{\omega}_k \mathbf{n}_L dl = 0, \quad k = 1, 2, \ldots, N \tag{3.2}$$

where $\boldsymbol{\omega}_k$ is a vector with components $\omega_{ki} = \omega_k l_{ki}$ $(i = 1, 2)$ and $\mathbf{n}_L$ is the unit vector of an outward normal to the contour L which bounds an arbitrary simply connected domain $V \subset G$.

The proof is by contradiction. Suppose that the kth family of fibres fits a certain curve S. Consequently, each trajectory touches the curve S at a certain point (see Fig. 1). From this family, we pick out two trajectories T_1 and T_2 which have points of contact S_1 and S_2 with S. On the trajectory T_2, we specify a point A_2 which differs from S_2 and, from it, we draw a curve S^* such that the tangents to it are perpendicular to the family of reinforcement trajectories. The point of intersection of the curve S^* with the trajectory T_1 is denoted by A_1 (all of these conditions can be satisfied if one selects two close contours T_1 and T_2). Condition (3.2) holds in the case of a piecewise-smooth contour $L = A_1S_1S_2A_2A_1$, and we rewrite this condition in the form

$$\oint_L \omega_k \cos(\alpha_k - \delta) dl = \int_{S_1S_2} \omega_k \cos(\alpha_k - \delta) dl + \int_{A_2A_1} \omega_k \cos(\alpha_k - \delta) dl = 0 \tag{3.3}$$

where δ is an angle defining the direction of the vector $\mathbf{n}_L$, and the integrals along the curvilinear segments A_1S_1 and S_2A_2 are equal to zero since the vectors are orthogonal to them. By virtue of the construction of the curve S^* in the segment $A_2A_1 \subset S^*$, $\alpha_k - \delta = \pi$ and, hence, when the condition $\omega_k > 0$ is satisfied in this segment, (3.3) can be rewritten in the form

$$\int_{S_1S_2} \omega_k \cos(\alpha_k - \delta) dl = \int_{A_2A_1} \omega_k dl = c > 0 \tag{3.4}$$

However, since the curve S^* is the envelope of the kth family of fibres, the inequality $\cos(\alpha_k - \delta) = 0$ holds in the curvilinear segment $S_2S_1 \subset S$. Consequently, in order that the integral on the left-hand side of (3.4) should be positive, an unbounded increase in the function ω_k as it approaches the curve S is required, and this means that the last condition of (1.6) will be violated close to the envelope of the family of reinforcement trajectories. The resulting contradiction therefore shows that the family of fibres of constant cross-section cannot have an envelope and, in particular, that the reinforcement trajectories of such fibres cannot approach along tangential directions to the contour of the structure.

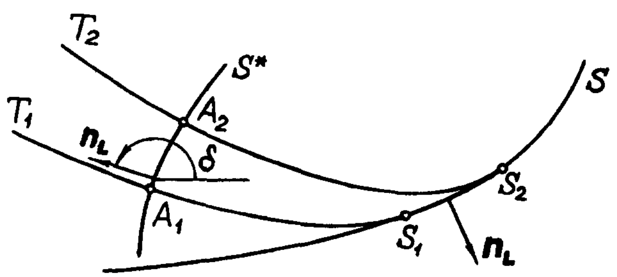

Fig. 1.

Now, suppose that the domain G occupied by the reinforced structure is doubly connected and bounded by an external contour Γ_1 and an internal contour Γ_2. We assume that fibres of the kth family enter the domain G in a certain curvilinear segment $A_1A_2 \subset \Gamma_1$ which is bounded by the points A_1 and A_2 and that these same fibres leave domain G in a curvilinear segment $S_1S_2 \subset \Gamma_2$ which is bounded by the points S_1 and S_2. Moreover, the points A_1 and S_1 belong to the same reinforcement trajectory, and the points A_2 and S_2 to another trajectory. In this case, equality (3.3) will hold again. We now start to contract the internal contour Γ_2 to a certain point when the length of the curvilinear segment S_1S_2 (together with the length of the contour Γ_2) will tend to zero. Since the integral along the segment A_2A_1 in (3.3) remains constant and is now zero, and the length of the segment S_1S_2 tends to zero, the function ω_k in this segment must increase without limit as the contour Γ_2 is contracted to a point, which violates the last condition of inequality (1.6). Consequently, if in the case of a doubly connected structure fibres of constant cross-section enter the structure on one contour and leave from the other, it is then impossible to obtain a reinforced simply connected structure by contracting the internal contour to a point.

In many RR problems in doubly connected structures such conditions for the stacking of the fibres are, in fact, satisfied. Hence, in the case of a doubly connected structure, which the fibres enter along the whole contour that bounds it, it is necessary to introduce a closed line of discontinuity of the solution through which the fibres would leave the reinforced subdomain, and within the domain which is bounded by this line of discontinuity in the solution, the structure of the material must be known (it is made of an isotropic material, for example). Such problems may be called rational reinforcement problems with a reinforcement where the mechanical characteristics are known.

The mathematical apparatus which has been presented above can be used to prove the following fact: fibres of constant cross-section belonging to one and the same family cannot intersect and cannot asymptotically approach one and the same curve, since, in this case, the last inequality of (1.6) will be violated.

The results obtained above, based on the condition for the cross-sections of the fibres to be constant, remain valid for any plane structures and plates which are reinforced with such fibres and not just for plane composite structures with rational stacking of the reinforcement.

We now consider the issue of the non-uniqueness of the solution of the RR problem. Actually, the static, kinematic and thermal boundary conditions (1.9), (1.10) and (2.6) are natural in problems of the mechanics of a deformable solid and are defined by the actual conditions under which the structure is used. The initial conditions for the reinforcement intensities (1.11) are "technological" conditions that specify the quantity of fibres of the kth family which are embedded in the structure in a given segment of the contour. The choice of the amount of embedded fibres is arbitrary to a certain extent. It only satisfies the constraints (1.6) and the conditions for the existence of the corresponding RR design. Consequently, an RR problem possesses functional arbitrariness associated with initial conditions (1.11) and the greater the number of families of fibres, the greater the number of these arbitrary factors. By varying initial conditions (1.11), it is possible to obtain whole "pencils" of solutions of an RR problem from which designs with certain properties can be selected such as, for example, with the minimum usage of reinforcing fibres, with the least intensity of stresses in the matrix, the least compliance, or those which area the most convenient from the point of view of their technological implementation. This means that it is possible to control rational designs. Furthermore, by virtue of the substantial non-linearity of the static and thermal boundary conditions in the functions α_k, ω_k an RR problem can have several solutions even in the case of fixed boundary conditions (1.11). All of this extends the "spectrum" of solutions of an RR problem even more, and the most practically achievable designs can be selected from this spectrum.

4. ANALYSIS OF SOME SOLUTIONS OF AN RR PROBLEM

In plane structures, the supporting elements are subject to comparable stress in areas which are differently orientated along the two directions and, from a practical point of view, it is quite satisfactory in plane RR problems to embed two families of uniformly stressed fibres in a structure. For this reason, we shall take $N = 2$ in the examples which are presented below.

We will initially consider the RR problem of an annular plate which is bounded by circles of radius r_0, r_1: $r_0 = 0.5r_1$. The internal contour is not under any load: $p_n = p_\tau = 0$ while a uniform normal load: $p_n = 0.08\sigma_1$, $p_\tau = 0$ is applied to the external contour. The distributed mass loads and thermal action are not taken into account: $b_1 = b_2 = \theta = 0$. The mechanical properties of the matrix and fibre materials are defined by the inequalities: $\sigma_1 = \sigma_2 > 0$, $E_1 = E_2$, $\lambda_1 = \lambda_2$, $\alpha_{a1} = \alpha_{a2}$, $E = 0.01E_1$, λ_1/λ_c

$= \alpha_{a1}/\alpha_c = 1.5$, $\nu = 0.25$, that is, the fibres are made from a single material. We choose the initial conditions for the intensity of the reinforcement in the external contour as: $\omega_{n1}(r_1) = \omega_{n2}(r_1) = 0.05$. A rational reinforcement design was obtained for these axially symmetric input data, and the reinforcement trajectories corresponding to it and a graph of the intensity of the stresses in the matrix ($\sigma_u(r)$) are shown in Figs 2 and 3.

This problem is of special interest for the following reasons. The internal contour is free from any load and the averaged stresses σ_{nn} normal to it will therefore be zero, while the stresses in the phases of the composition constituting σ_{nn} are non-zero in this case. Actually, it can be seen from Fig. 2 that the reinforcement trajectories of the uniformly stressed fibres approach the internal contour along lines which are close to tangents (they cannot be tangents as was shown in Section 3). The closeness of the trajectories to tangents substantially reduces the contribution of the stresses in the fibres to the averaged stresses σ_{nn}, but, by virtue of the fact that these trajectories differ somewhat from tangents, this contribution is not equal to zero and hence, when $r = r_0$, it must be compensated by stresses in the matrix. On the other hand, as was shown in Section 3, the closeness of the reinforcement trajectories to lines which are tangents to the internal contour leads to a sharp increase in the functions ω_1, ω_2 when $r \to r_0 + 0$, and this implies a sharp decrease in the intensity of the interlaying of the binder $a = (1 - \omega_1 - \omega_2)$ when $r \to r_0 + 0$. The small value of the ratio E/E_1 and the pronounced decrease in a leads to a state of affairs where the deformations close to the internal contour must reach values with large moduli in order to compensate in σ_{nn} for the contribution from the stresses in the fibres. T his fact is clearly reflected in the graph of the function $\sigma_u(r)$ (Fig. 3): the intensity of the stresses in the

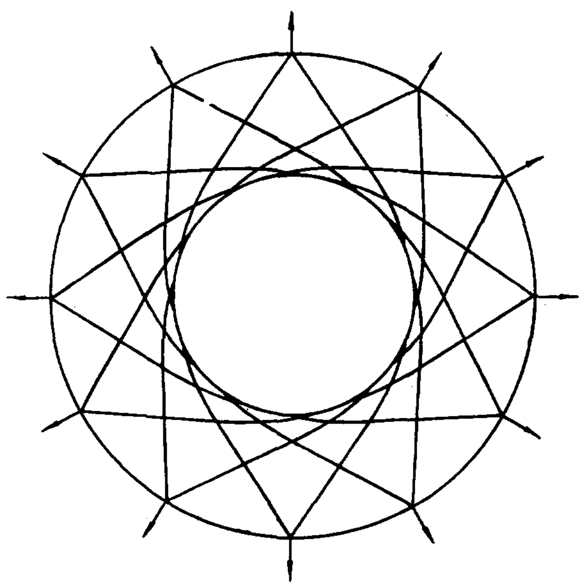

Fig. 2.

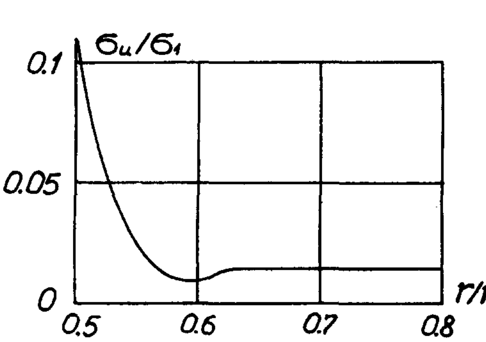

Fig. 3.

matrix increases rapidly when $r \to r_0 + 0$, exceeding the values of $\sigma_u(r)$ at points of the structure which are distant from the internal contour by an order of magnitude. (For comparison, it is pertinent to point out here that, in the case of an isotropic plate with the same dimensions and the same load, the ratio of the intensities of the stresses in the internal contour to the same quantity in the external contour is equal to $\sigma_u(r_0)/\sigma_u(r_1) = 1.84$.)

If an asymptotic analysis is carried out on system (2.1)–(2.6), taking the ratio E/E_1 as a small parameter, then, when there are no distributed mass loads, the first approximations for the reinforcement trajectories will be straight lines. The reinforcement trajectories shown in Fig. 2 are in accordance with this conclusion since they are close to straight lines in almost the whole of the structure. Pronounced curvature in them is only observed in the neighbourhood of the internal contour which is clearly attributable to the large deformations in this neighbourhood.

The example presented above clearly reflects the fact that the requirement that the conditions for the uniform stressing of the high-modulus fibres throughout the whole of a structure with a specified level of stresses must be satisfied is very rigorous. This requirement, on the one hand, cannot be achieved in a number of structures even from a mathematical point of view and, on the other hand, can lead to RR projects in which the binder will be fractured. Consequently, in order to moderate conditions for rational planning, it is advisable to consider RR problems with discontinuous solutions when the fibres in different subdomains of a structure are made of different materials, or to introduce reinforcements where the structure of the reinforcement material is already known. For instance, in the above example, one should introduce an annular reinforcement made of an isotropic material, for example, in the internal contour.

We will now demonstrate the effect of a temperature field on the structure of a rational reinforcement using the following example. A plane structure is bounded by two contours which are specified by equalities in a polar system of coordinates. The internal contour $r_0(\varphi) = D(0.5 - 0.05\cos(2\varphi - \pi/2))$ and the external contour $r(\varphi) = D(1 + 0.08\cos 2\varphi)$, where D is the characteristic dimension of the structure. The structure is clamped along the external contour and a uniform normal load is applied to the internal contour: $p_n = 0.5\sigma_1$, $p_\tau = 0$. Distributed mass loads are ignored: $b_1 = b_2 = 0$. The mechanical properties of the binder and fibre materials are the same as in the preceding example. We choose the initial conditions for the reinforcement intensities in the internal contour as: $\omega_{n1}(\eta) = \omega_{n2}(\varphi) = 0.4$.

An RR design without allowing for thermal action was obtained for these input data. The structure of the reinforcement, corresponding to it, is shown in Fig. 4 by the dashed lines, and the reinforcement trajectories do not differ visually from straight lines.

Suppose that there is a heat flux across the end surfaces of the structure, that the intensity of the internal thermal sources is equal to zero and that a dimensionless value of the temperature $E_1\alpha_c\theta_0/\sigma_1 = 6$ is specified at a certain point of the contour. According to the results in Section 3, for such thermal action, the temperature field will be constant and equal to θ_0. Consequently, the heat conduction problem in this example can be considered to have been solved independently of the solution of the RR problem.

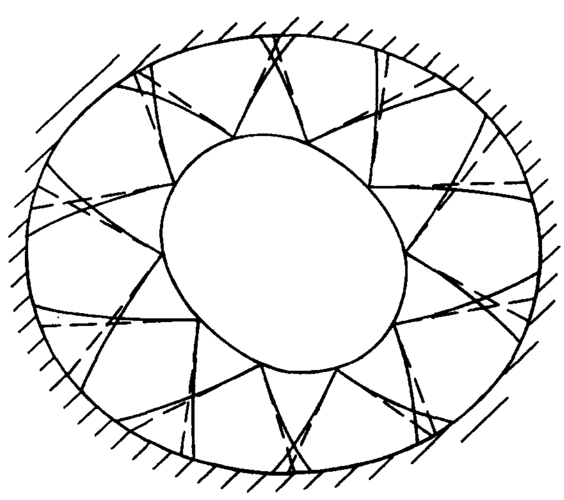

Fig. 4.

The structure of the rational reinforcement corresponding to such uniform heating is represented by the solid lines in Fig. 4. A comparison of the RR designs presented in Fig. 4 shows that the heating of a structure leads to a concentration of the reinforcement trajectories and to the appearance of their clearly defined curvature. Moreover, a thermal action has a substantial effect on the stress–strain state of a structure. Thus, the greatest value of the stress intensity in the binder in a heated structure is 11.8 times greater than in a temperature-free design.

Calculations showed that similar reinforcement designs can correspond to different types of heating of the structure. If the coefficients of linear thermal expansion of the fibres are greater than the corresponding coefficient for the binder, then the RR trajectories become concentrated when the sign of the temperature is identical to the sign of the stress in fibres of equal strength and the fibres open out when these signs are different (the pattern changes to the opposite pattern if the above-mentioned coefficients of the fibres are smaller than the coefficient of the binder). The results obtained for different thermal boundary conditions confirm the fact that thermal action has a substantial effect on the RR structure and on the stress-strain state of a structure and that this action cannot be neglected.

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